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On quantum probabilities in the measurement of both complex risk and the expected loss in a loss event of a co-insured proportion insurance business

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Abstract: This paper discusses the effect of probability density in contemporary actuarial risk analysis by starting with statistical mechanics tools while concentrating on singularity potential. The natural characteristics of the probability density in actuarial statistics is shown to depend on the precise description of the quantum measurement of insurance risk problem as it occurs in general co-insured contracts. In general insurance contracts, coinsurance is a means of managing potential risk where the insured and the underwriter agree for payment of covered losses in a predetermined proportion after satisfying deductible conditions making the insured to be jointly and proportionally liable for losses. The scheme holder will be jointly and proportionately liable for losses provided he insures properties smaller than the legally required proportion of its full value but where the insured does not want to self-insure, the coinsurance clauses automatically bind policyholder to preserve sufficient coverage against insured risk. The influence of structural properties of dirac-delta on coinsurance clause is studied in this paper. The aim is to obtain model for the expected loss in a loss event. In order to justify a sound mathematical basis in formulating coinsurance model, the general properties of dirac delta are first examined in respect of severity probability density. Furthermore, some theorems were stated and proved as part of our contributions. From the result obtained, the variance tends to zero when the co-insurance factor equals the j^{th} probability of the j^{th} risk.

Keywords: statistical mechanics, probability density, singularity potential, coinsurance, deductible

1. Introduction to Singularity Functions

Dirac-delta function is used in many disciplines of science particularly in statistical mechanics where it is usually applied to the analysis of divergence of asymptotic function. Following (Khuri 2004; Onural, 2006; Mohammed 2011), the incongruous characteristics of dirac-delta function is that it does not seem to be categorized as a real valued function by definition but a further extension of Kronecker delta to continuous function. In view of (Pazman & Pronzato, 1996; Zhang, 2018), the dirac-delta function $\delta(x)$ is not a function of x with a defined value at every point in the domain and hence it is usually described as a distribution function. $\delta(x)$ is applied to obtain definite asymptotic nomenclature to address clearly, concepts which are associated with a certain type of infinity. In quantum theory, $\delta(x)$ is associated with the fact that the eigen function connected with their corresponding eigen value in the continuum is non-normalisable.

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The dirac-delta function is described by associating rules in integrating its product and a continuous function. Following (Dirac, 1985; Das & Melissinos, 1986), Suppose that a quantum state $|k\rangle = k_1|y_1\rangle + k_2|y_2\rangle + k_3|y_3\rangle + k_4|y_4\rangle + \ldots k_m|y_m\rangle$ is defined where $|y_j\rangle$ are the orthonormal basis. Then by the orthonormality condition, we have $\langle y_u|y_v\rangle = \delta_{uv} = \begin{cases} 0 & u \neq v \\ 1 & u = v \end{cases}$

The probability of obtaining the state $|k\rangle$ in the base state $|y_j\rangle$ is $\rho_u = \langle y_u|k\rangle$. By reason of orthonormality condition in view of (Dirac, 1985; Das, & Melissinos, 1986), $\rho_u = k_u$. Thus the Expansion coefficients $k_u; u = 1,2,3,\ldots,m$ defining a state $|k\rangle = k_1|y_1\rangle + k_2|y_2\rangle + k_3|y_3\rangle + k_4|y_4\rangle + \ldots k_m|y_m\rangle$ in a specified framework is the probability for obtaining the arbitrary state in the corresponding basis state.

Consequently $|k\rangle = \sum_{u=1}^{m} \langle y_u|k\rangle|y_u\rangle = \sum_{u=1}^{m} \rho_u|y_u\rangle$. Let $|\varsigma\rangle$ be the state of a body in a straight line. By this definition, the basis state $|y\rangle$ means that the body is at state y and it is continuous. Similar to the previous definition, $|\varsigma\rangle = k_1|y_1\rangle + k_2|y_2\rangle + k_3|y_3\rangle + k_4|y_4\rangle + \ldots k_m|y_m\rangle$. Because $|y\rangle$ is continuous, it is then defined by integrals $|y\rangle = \int h(y)|y\rangle dy$ where $h(y)$ is the probability of obtaining the position of the body at y. At a different point $\bar{y}$, the probability of obtaining the particle is $\int h(y)|\bar{y}\rangle dy = \langle \bar{y}|\varsigma\rangle = h(\bar{y})$. As $\bar{y} \to 0; \langle \bar{y}|\varsigma\rangle \to \langle 0|\varsigma\rangle$ and $\langle 0|\varsigma\rangle$ approaches a function of y defined by $\eta(y)$ such that $\int h(y)\eta(y)dy = h(0)$. The integral does not depend on which values $h(y)$ takes for values of y other than 0 and hence $\eta(y) \to 0$ for all values of y except 0. However, if $\eta(y) \to 0$ everywhere, then $\int h(y)\eta(y)dy \to 0$ and $\int h(y)\eta(y)dy = h(0)$ will not be valid, consequently there no such function $\eta(y)$

Following (Ogungbenle *et al.*, 2021), we argued that in the determination of the limiting values, there is rarely a mathematical function that meets the requirement in the integral.

Following the arguments in (Salasnich, 2014), $\int_{-\infty}^{\infty} \delta(s)ds = \lim_{\eta \to 0^+} \int_{-\infty}^{\infty} \delta(\eta,s)ds = \lim_{\eta \to 0^+} \int_{-b}^{b} \delta(\eta,s)ds, b \to \infty$

$$\lim_{\eta \to 0} \delta(\eta,s) = \begin{cases} \infty, \text{if}, x = 0 \\ 0, \text{if}, x \neq 0 \end{cases}$$

$\int_{-\infty}^{\infty} \delta(s)ds = 1$, the delta function has been normalized to unity

Probability is usually valid on the real line and the integral of a probability density on the real line is 1. In a collection of insurance risks, it is possible to identify a set of risk measures for the associated risk and a potentially possible actuarial model for the insurance risks in such a way that every insurance risks is assigned by the model to a collection of random variables describing the insurance risk variables connected with the risk. In general insurance business, risk is described by a random variable whose analysis is carried out through probability indicators such as expectation and variance function. Following (Bass *et al*, 2020), the co-insurance problem is usually dealt with in terms of probability risk theory. Coinsurance is defined as the apportionment of severities between the policyholder and underwriter where the underwriter indemnifies only a proportion of every insured loss equal to the coinsurance apportionment ratio which applies subsequent to satisfying deductible modifications and policy restrictions. Insurance to value subsists provided the property is insured to the required level assumed in the premium rate computations. Preserving the insurance to value specifies the rationale behind maintaining cover on the insured at a threshold commensurate to the assumptions within the actuarial premium rating. In (Bass *et al*, 2020), coinsurance clause applies when the contract has fallen short of the required level of insurance cover based on the coinsurance clauses defined by the underwriting process and consequently preserves the actuarial balances in premium rating determination where the policyholder has underinsured. In the event of loss falling below the coinsurance conditions, then the coinsurer will be penalized by an amount $\beta > 0$

$$\beta = \begin{cases} l - i & for \quad l \leq S \\ S - i & for \quad S < l \leq \alpha\rho \\ 0 & for \quad l \geq \alpha\rho \end{cases}$$

by which the indemnity for a loss is reduced by the terms and conditions of the coinsurance clauses where S is the sum insured, i is the indemnity payment to the scheme holder per policy period, l is the value of the loss, ρ is the value of the property and α is the co-insurance percentage but where the loss is greater than the coinsurance conditions, then the indemnity i is limited to the face value of the contract and coinsurance penalty does not apply.

If $f_L(l)$ is the probability of a loss, then the expected value of the indemnity given that the value of a loss lies between 0 and S inclusive is given by $E\left(i / 0 \leq l \leq S\right) = \dfrac{\int_0^S l \times f_L(l)\,dl}{\int_0^S f_L(l)\,dl}$ .We note that where coinsurance condition is not written in the contractual agreement, then it has no responsibility for the insured who keeps a certain level of cover that is approximately equivalent to the sum insured on the property. Furthermore, property coinsurance places an obligation on the scheme holder to maintain a definite amount of insurance in force on the property insured otherwise, the insured will be penalized where a loss occurs. In view of the burden on the insured to keep adequate cover, the underwriter writing the clause is duty bound to state clearly the policy conditions. When an underwriter is compelled to indemnify monetary values higher than the policy value, then they will not be opportune to recover the premiums on all schemes that have been analogously underinsured to compensate hence it is advisable for an underwriter to make sure that the insured keeps the required level of insurance cover within the actuarial premium computations. This paper demonstrates how dirac-delta has been applied to formulate actuarial density of an insurance risk. Thus, in explaining a unified ground of applying quantum functions to investigate the behavior of insurance risk functions, the singularity potential method was applied to investigate expected loss in a coinsured business regarding claim severities, the variance function and complex risk. The rationale behind employing singularity potential function, is characterized in its ease to permit alternative method to attain analytically useful models for coinsured business severity.

2. Material and Methods

We follow methodology in nomenclature and methodologies in (Chakraborty, 2008) and in particular (Ogungbenle *et al*, 2020; Ogungbenle, *et al*., 2021) where singularity potential was initiated to find moments of risk functions. Consider

$$x_a < t_0 < x_b, \int_{x_a}^{x_b} \delta(x-t_0)\,g(x)\,dx = \int_{x_a}^{x_b} \delta(x-t_0)\,g(t_0)\,dx = g(t_0)\int_{x_a}^{x_b} \delta(x-t_0)\,dx \qquad (1)$$

$$\int_{x_a}^{x_b} \delta(x-t_0)\,g(x)\,dx = g(t_0) \qquad (2)$$

for any continuous $g(x)$ defined on real line at a x and some point $t_0 \in \mathbb{R}$

If $t_0 = 0$, we have $\int_{x_a}^{x_b} \delta(x)\,g(x)\,dx = g(0)$ (3)

Define $H(x-a) = h_a(x)$ (4)

$\int_{\mathbb{R}} f(x)\,h_b(x)\,dx = \int_{\mathbb{R}} f(x)\,dx$ for some integrable function $f(x)$ on $(-\infty,\infty)$.

When $a = 0$, equation (4) above can be expressed as $h_0(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geq 0 \end{cases}$ (4a)

However, where the mean value at point 0 is taken we have $h_0(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{1}{2} & \text{for} \quad x \equiv 0 \\ 1 & \text{for} \quad x > 0 \end{cases}$ (4b)

If we define the indicator function $I_{x\in\Omega} = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geq 0 \end{cases}$ (4c)

then we can find a relationship $h_0(x) = I_{0<x<\infty}$

The probability density is defined by $f_X(\theta) = E_x\left(I_{0<x\leq\theta}\right) = E_x\left(H(\theta - x)\right) = h_x(\theta)$ (4d)

then by extension, we have $I_{(A-y)<B\leq(A-x)} \equiv I_{x+B\leq A<y+B}$. If $f(x)$ is continuous, then

$$\int_{-\infty}^{\infty} f(x)h_0'(x)\,dx = \left[\int_{-\infty}^{\infty} f'(x)h_0(x)\,dx - f(x)h_0'(x)\right]_{-\infty}^{\infty} = \int_{0}^{\infty} f'(x)\,dx \to f(0) \qquad (4e)$$

Following (Sastry 2009; Zhang, 2018), because $H(x)$ is a unit step function, the equivalence holds

$$\delta(x-a)\,dx = dh_a(x) \qquad (5)$$

and difference at two different arguments in Heavy side is defined as follows

$$H(x-x_a) - H(x-x_a-\varsigma) = \varsigma\delta_\varsigma(x-x_a) \qquad (6)$$

$$\Rightarrow \varsigma\delta_\varsigma(x-x_a) = h_{x_a}(x) - h_{x_a+\varsigma}(x) \qquad (7)$$

assuming $x_a = 0$ and taking limit as $\varsigma \to o(1)$, where $o(1)$ is a function which vanishes, then we have that $\lim_{\varsigma\to 0}\delta_\varsigma(x) = \lim_{\varsigma\to 0}\frac{h_0(x)-h_\varsigma(x)}{\varsigma} = \frac{dH}{dx}$ (8)

If $f_X(x)$ is a probability function, then the function $\left[h_{x_a}(x) - h_{x_b}(x)\right] f_X(x)$ means the probability function $f_X(x)$ is 1 between $x = x_a$ and $x = x_b$ but $f_X(x)$ will be 0 when $x < x_a$ or $x > x_b$. By the same reasoning, $\left[h_{x_a}(x)\right] f_X(x)$ means $f_X(x)$ is 1 as $x > x_a$ but $f_X(x)$ is 0 when $x < x_a$

Let $f(x) = \begin{cases} y_1(x) & for \quad 0 < x < b \\ y_2(x) & for \quad x > b \end{cases}$, then $f(x)$ can be expressed as follows

$$\begin{cases} y_1(x) & for \quad 0 < x < b \\ y_2(x) & for \quad x > b \end{cases} = y_1(x)\left[h_0(x) - h_a(x)\right] + y_2(x)\left[h_a(x)\right] \qquad (8a)$$

$$f(x) = y_1(x)\left[H(x-0) - H(x-a)\right] + y_2(x)\left[H(x-a)\right] \qquad (8b)$$

Following (Chakraborty, 2008; Ogungbenle *et al*, 2020; Ogungbenle *et al*, 2021), we let $G_X(x)$ be the cumulative distribution function of an insurance risk X characterized as follows

$$dG_X(x) = g_X(x)\,dx \qquad (9)$$

Define $G_X(x) = \sum_{x_i\in\Omega_X} P(x_i)h_{x_i}(x)$, where Ω_X is the support of X.

$$\frac{dG_X(x)}{dx} = \frac{d}{dx}\left[\sum_{x_i\in\Omega_X} P(x_i)h_{x_i}(x)\right] = \left[\sum_{x_i\in\Omega_X} P(x_i)\frac{d}{dx}\left[h_{x_i}(x)\right]\right], \qquad (10)$$

so that the probability density function is obtained as

$$\frac{dG_X(x)}{dx} = g_X(x) = \sum_{x_i\in\Omega_X} P(x_i)\,\delta(x-x_i) \qquad (11)$$

where $\Omega_x = \{x_1, x_2, x_3, \ldots\}$ and $P(x_i)$ are the probability mass points. The moments of the insured risk X are as follows

$$E\left(X\right)=\int_{-\infty}^{\infty} sf_X\left(s\right)d=\int_{s=0}^{\infty}\left[\int_{y=0}^{s} dy\right]dF_X\left(s\right)=\int_{x=0}^{\infty}\int_{t=x}^{\infty} dF_X\left(s\right)dx=\int_{0}^{\infty} S_X\left(s\right)dx \tag{12}$$

$$E\left(X\right)=\int_{-\infty}^{\infty} xg_X\left(x\right)dx=\int_{-\infty}^{\infty}\left[x\sum_{x_i\in\Omega_X} P\left(x_i\right)\delta\left(x-x_i\right)\right]dx \tag{13}$$

$$E\left(X\right)=\int_{-\infty}^{\infty}\left[x\sum_{x_i\in\Omega_X} P\left(x_i\right)\delta\left(x-x_i\right)\right]dx=\left[\sum_{x_i\in\Omega_X} P\left(x_i\right)\int_{-\infty}^{\infty} x\delta\left(x-x_i\right)dx\right] \tag{14}$$

$$E\left(X\right)=\left[\sum_{x_i\in\Omega_X} x_iP\left(x_i\right)\int_{-\infty}^{\infty}\delta\left(x-x_i\right)dx\right]=\left[\sum_{x_i\in\Omega_X} x_iP\left(x_i\right)\right] \tag{15}$$

$$E\left(X^m\right)=\int_{-\infty}^{\infty} x^m g_X\left(x\right)dx=\int_{-\infty}^{\infty}\left[\sum_{x_i\in\Omega_X} P\left(x_i\right)\delta\left(x-x_i\right)\right]x^m dx \tag{16}$$

$$E\left(X^m\right)=\left[\sum_{x_i\in\Omega_X} P\left(x_i\right)\int_{-\infty}^{\infty} x^m\delta\left(x-x_i\right)dx\right]=\sum_{x_i\in\Omega_X} x_i^{\,m}P\left(x_i\right) \tag{17}$$

Clearly, $Var\left(X\right)=E\left(X^2\right)-\left(E\left(X\right)\right)^2=\left[\sum_{x_i\in\Omega_X} x_i^{\,2}P\left(x_i\right)\right]-\left[\sum_{x_i\in\Omega_X} x_iP\left(x_i\right)\right]^2$ (18)

2.1 Theorem 1

Suppose Z is an insurable risk and let $h:\mathbb{R}^{+}\to\mathbb{R}^{+}$ such that $\frac{d^r h}{du^r}\geq 0$; for $0\leq\left\langle Z^r\right\rangle<\infty$ (19)

$$\int_{0}^{\infty}\frac{\left(z-u\right)_{+}^{r-1}}{\left(r-1\right)!}\frac{d^r h}{du^r}du=h\left(z\right)-\sum_{n=0}^{r-1}\left[\frac{\left\langle Z^n\right\rangle\left[\frac{d^n h}{du^n}\right]_{u=0}}{\int_{0}^{\infty}\frac{\left(\log_e z\right)^n}{z^2}dz}\right] \tag{20}$$

Proof

Let $\left\langle . \right\rangle$ be the average value function. $\lim_{c\to\infty}\left(Z-c\right)_{+}=\int_{c}^{\infty} S_Z\left(\chi\right)d\chi=0$ (21)

$$\left\langle z\right\rangle=\int_{0}^{\infty}\zeta dF_Z\left(\zeta\right)=\int_{\zeta=0}^{\infty}\int_{z=0}^{\zeta} dz dF_Z\left(\zeta\right)=\int_{z=0}^{\infty}\int_{\zeta=z}^{\infty} dF_Z\left(\zeta\right)dz=\int_{0}^{\infty} S_Z\left(z\right)dz<\infty \tag{22}$$

Hence $\left\langle z^r\right\rangle<\infty,\left\langle z^r\right\rangle>0, r\varepsilon\mathbb{R}^{+}$ (23)

Recall $n! = \int_1^\infty \frac{(\log_e z)^n}{z^2} dz$ (24)

$$h(z) = \frac{h^{(0)}(0)}{0!} z^0 + \frac{h^{(1)}(0)}{1!} z^1 + \frac{h^{(2)}(0)}{2!} z^2 + \frac{h^{(3)}(0)}{3!} z^3 + \ldots + \frac{h^{(r)}(0)}{r!} z^r + \frac{h^{(r)}(u)(z-u)^{r-1}}{(r-1)!} + \int_0^\infty \frac{h^{(r)}(u)(z-u)^{r-1}}{(r-1)!} du \tag{25}$$

$$\langle h(z) \rangle = \frac{h^{(0)}(0)}{0!} \langle z^0 \rangle + \frac{h^{(1)}(0)}{1!} \langle z^1 \rangle + \frac{h^{(2)}(0)}{2!} \langle z^2 \rangle + \frac{h^{(3)}(0)}{3!} \langle z^3 \rangle + \ldots + \frac{h^{(r)}(0)}{r!} \langle z^r \rangle + \int_0^\infty \frac{h^{(r)}(u) \langle (z-u)^{r-1} \rangle}{(r-1)!} du = \sum_{n=0}^{r} \frac{h^{(n)}(0)}{n!} \mu_n + \int_0^\infty \frac{h^{(r)}(u) \langle (z-u)^{r-1} \rangle}{(r-1)!} du \tag{26}$$

$$\int_0^\infty \frac{h^{(r)}(u) \langle (z-u)_+^{r-1} \rangle}{(r-1)!} du = \int_0^\infty \frac{h^{(r)}(u) \int_u^\infty S_Z^{(r)}(\zeta) d\zeta}{(r-1)!} du = \tag{27}$$

$$\int_{u=0}^\infty \int_{\zeta=u}^\infty \frac{S_Z^{(r)}(\zeta) d\zeta h^{(r)}(u) du}{(r-1)!} = \int_0^\infty \frac{h^{(r)}(u) \langle (z-u)^{r-1} \rangle}{(r-1)!} du$$

$$\int_0^\infty \frac{h^{(r)}(u) \langle (z-u)_+^{r-1} \rangle}{(r-1)!} du = \int_0^\infty \frac{h^{(r)}(u) \langle (z-u)^{r-1} \rangle}{(r-1)!} du \tag{28}$$

$$\int_0^\infty \frac{\langle (z-u)_+^{r-1} \rangle}{\left[\int_0^\infty \frac{(\log_e \psi)^{n-1}}{\psi^2} d\psi \right]} \frac{d^r h}{du^r} du = \langle h(z) \rangle - \sum_{n=0}^{r-1} \left[\frac{\langle z^n \rangle}{\int_0^\infty \frac{(\log_e \psi)^n}{\psi^2} d\psi} \times \left. \frac{d^n h}{du^n} \right|_{u=0} \right] \tag{29}$$

3. Characteristics Function

Let X be a complex risk with density $g_X(x)$ and if $i = \sqrt{-1}$ then the Fourier transform of the probability function is defined by $\theta_X(s) = E(e^{isx}) = \int_{-\infty}^{\infty} e^{isx} g_X(x) dx$. (30)

This equivalently describes the characteristic function of X .

This definition is now extended to k dimensional Euclidean space $\mathbb{R}^k$

Let $x_1, x_2, x_3, ...x_k$ be a set of insurance k independent risks and their densities defined by $\{g_i(x_i)\}_{i=1}^k$.

Suppose the characteristic function of individual risk x_i is $\theta_1(s), \theta_2(s), \theta_3(s), ..., \theta_k(s)$.

$$\text{Let } h = \sum_1^k x_i \tag{31}$$

$$\theta_h(s) = \int_{\mathbb{R}} ... \int_{\mathbb{R}} \left(e^{is\sum_1^k x_i} \right) g_1(x_1) g_2(x_2) g_3(x_3) g_4(x_4) ... g_k(x_k) ... dx_1 dx_2 dx_3 .... dx_k \tag{32}$$

$$\theta_h(s) = \left(\int_{\mathbb{R}} e^{isx_1} g_1(x_1) dx_1 \right) \left(\int_{\mathbb{R}} e^{isx_2} g_2(x_2) dx_2 \right) \left(\int_{\mathbb{R}} e^{isx_3} g_3(x_3) dx_3 \right) .... \left(\int_{\mathbb{R}} e^{isx_n} g_n(x_n) dx_n \right) \tag{33}$$

$$\theta_h(s) = E\left(e^{isx_1}\right) E\left(e^{isx_2}\right) E\left(e^{isx_3}\right) ... E\left(e^{isx_{n-2}}\right) E\left(e^{isx_{k-1}}\right) E\left(e^{isx_k}\right) \tag{34}$$

$$\theta_h(s) = \theta_1(s) \theta_2(s) \theta_3(s) ... \theta_k(s)$$

The function $\theta_X(s)$ wholly specifies the distribution of random risk X to the extent that when $\theta_X(s) = \theta_Y(s)$, then X and Y will be identically distributed.

$$\frac{d}{ds}\theta_h(s) = \frac{d}{ds}\int_{-\infty}^{\infty} e^{ish} g_X(h) dh = \int_{-\infty}^{\infty} \frac{\partial}{\partial s} e^{ish} g_X(h) dh = \int_{-\infty}^{\infty} ihe^{ish} g_X(h) dh =$$
$$ie^{ish} \int_{-\infty}^{\infty} hg_X(h) dh = ie^{ish} E(h) \tag{35}$$

$$\frac{d^2}{ds^2}\theta_h(s) = \frac{d^2}{ds^2}\int_{-\infty}^{\infty} e^{ish} g_X(h) dh = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial s^2} e^{ish} g_X(h) dh =$$
$$\int_{-\infty}^{\infty} i^2 h^2 e^{ish} g_X(h) dh = i^2 e^{ish} E\left(h^2\right) \tag{36}$$

$$\frac{d^r}{ds^r}\theta_h(s) = \frac{d^r}{ds^r}\int_{-\infty}^{\infty} e^{ish} g_X(h) dh = i^r e^{ish} E\left(h^r\right) \tag{37}$$

Now

$$\theta_h(ks) = \sum_{x \in \Omega_x} \left[(\cos sh) g_X(h) + i(\sin ksh) g_X(h) \right]$$
$$= \sum_{h \in \Omega_h} \left[(\cos ksh) g_X(h) \right] + i \sum_{h \in \Omega_h} \left[(\sin ksh) g_X(h) \right] \tag{38}$$

When the summation is replaced by the integral

$$\theta_h(ks) = E\left(e^{iksh}\right) = \int_{-\infty}^{\infty} \left[(\cos ksh) g_X(h) + i(\sin ksh) g_X(h) \right] dh \tag{39}$$

$$\theta_h(ks)=\int_{-\infty}^{\infty}\left[(\cos ksh)\,g_X(h)\right]dh+\int_{-\infty}^{\infty}\left[i(\sin ksh)\,g_X(h)\right]dh \tag{40}$$

$$\theta_h(ks)=\int_{-\infty}^{\infty}\left\{(\cos ksh)\sum_{j=1}^{k}P_j\delta\left(h-x_j^{*}\right)+i(\sin ksh)\sum_{j=1}^{k}P_j\delta\left(h-x_j^{*}\right)\right\}dh \tag{41}$$

$$\theta_h(ks)=\int_{-\infty}^{\infty}\left[(\cos ksh)\sum_{j=1}^{k}P_j\delta\left(h-x_j^{*}\right)+i(\sin ksh)\sum_{j=1}^{h}P_j\delta\left(h-x_j^{*}\right)\right]dh \tag{42}$$

$$\theta_h(ks)=\sum_{j=1}^{k}\int_{-\infty}^{\infty}(\cos ksh)P_j\delta\left(h-x_j^{*}\right)dh+i\sum_{j=1}^{k}\int_{-\infty}^{\infty}(\sin ksh)P_j\delta\left(h-x_j^{*}\right)dh \tag{43}$$

$$\theta_h(ks)=\sum_{j=1}^{k}P_j\int_{-\infty}^{\infty}(\cos ksh)\delta\left(h-x_j^{*}\right)dh+i\sum_{j=1}^{k}P_j\int_{-\infty}^{\infty}(\sin ksh)\delta\left(h-x_j^{*}\right)dh \tag{44}$$

$$\theta_h(ks)=\sum_{j=1}^{k}P_j\cos ksx_j^{*}+i\sum_{j=1}^{k}P_j\sin ksx_j^{*} \tag{45}$$

From equation (39) we define,

$$C_H(U)=\int_{-\infty}^{\infty}\cos(Uh)\,f_H(h)dh$$
$$S_H(U)=\int_{-\infty}^{\infty}\sin(Uh)\,f_H(h)dh \tag{46}$$

then $\beta_H(U)=\int_{-\infty}^{\infty}\cos(Uh)\,f_H(h)dh-\int_{-\infty}^{\infty}i\sin(Uh)\,f_H(h)dh$ (47)

$$\left|\beta_H(U)\right|^2=\left(\int\cos(Uh)\,f_H(h)\,dh\right)^2+\left(\int\sin(Uh)\,f_H(h)\,dh\right)^2 \tag{48}$$

$$\left|\beta_H(U)\right|^2=\left(\int_0^{\infty}\cos(Uh)\,f_H(h)dh\right)^2+\left(\int_0^{\infty}\sin(Uh)\,f_H(h)dh\right)^2 \tag{49}$$

$$\left|\beta_H(U)\right|^2=\int_0^{\infty}\int_0^{\infty}f_H(h)\,f_K(k)\left(\cos uh\cos uk+\sin uh\sin uk\right)dhdk \tag{50}$$

$$\left|\beta_H(U)\right|^2=\int_0^{\infty}\int_0^{\infty}f_H(h)\,f_K(k)\cos(uh-uk)dhdk\text{ , since }\left|\cos(uh-uk)\right|\le 1 \tag{51}$$

$$\left|\beta_H(U)\right|^2=\int_0^{\infty}\int_0^{\infty}f_H(h)\,f_K(k)\cos(uh-uk)dhdk\le\int_0^{\infty}\int_0^{\infty}f_H(h)\,f_K(k)dhdk=\int_0^{\infty}\delta(s)ds \tag{52}$$

In the case where there is a linear combination of insurance risks R_a and R_b such that $X = R_a + R_b$ The probability density function $f_X(x)$ of X applying dirac-delta function is

$$f_X(x) = \iint_R g(r_a, r_b)\delta(x - r_a - r_b)\,dr_a dr_b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - r_a - r_b)\, g(r_a, r_b)\,dr_a dr_b \quad (53)$$

$$f_X(x) = \int_{-\infty}^{\infty} dr_b \int_{-\infty}^{\infty} \delta(r_b - x + r_a)\, g(r_a, r_b)\,dr_a = \int_{-\infty}^{\infty} dr_b \int_{-\infty}^{\infty} \delta\left(r_a - (x - r_b)\right) g(r_a, r_b)\,dr_a \quad (54)$$

$$\int_{-\infty}^{\infty} g\left((x - r_b), r_b\right) dr_b = f_X(x) \quad (55)$$

Let $h(\mathrm{x})$ be a continuously smooth and integrable function of a random risk, then from (1), we have

$$h(\mathrm{s}_0) = \int_{-\infty}^{\infty} \delta(\mathrm{s} - \mathrm{s}_0)\, h(\mathrm{s})\,\mathrm{ds} \quad (56)$$

$$h(\mathrm{s}_0) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iU(s-t)} dU \right) h(\mathrm{s})\,\mathrm{ds} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iUt} \int_{-\infty}^{\infty} e^{iUs} dU h(\mathrm{s})\,\mathrm{ds} \quad (57)$$

$$\delta(s - t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iU(s-t)} dU \quad (58)$$

Thus when $t = 0$, then we have

$$\delta(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iUs} dU \quad (59)$$

The fast Fourier transform FT of $h(.)$ is defined as

$$FT(U) = \int_{-\infty}^{\infty} e^{iUs} h(\mathrm{s})\, ds \text{ so that } h(\mathrm{t}) = \int_{-\infty}^{\infty} e^{-iUt} FT(U)\, dU \quad (60)$$

The characteristic function describes the Fourier transform of the probability density function of a random risk S such that $FT(V) = \int_{-\infty}^{\infty} e^{iVs} \mathrm{f}(\mathrm{s})\, ds$. The function $\mathrm{f}(\mathrm{s})$ defines the final pay-off to a unit linked insurance which is maturing at time s.

Suppose δ is locally integrable over $[-\beta, \beta]$, the mean value $M_V(\delta)$ of $\delta(s)$ is given as

$$2\beta M_V(\delta) = \lim_{\beta \to \lambda} \int_{-\beta}^{\beta} \delta(s)\, ds \quad (61)$$

If $\delta(s + k) = \delta(s)$ for all s and k is the period, then it follows that

$$kM_V(\delta) = \int_0^k \delta(s)\, ds \quad (62)$$

If C is a union of a collection of intervals such that the length of every interval is a multiple of the fundamental period of δ and $\theta = \sum_{i=1}^{n} \alpha_i \theta_i$ a linear combination of characteristic function θ_i, then

$$\int_{-\infty}^{\infty} \theta(s) ds = \frac{k \int_{-\infty}^{\infty} \theta(s) \delta(s) ds}{\left[\int_{0}^{k} \delta(s) ds \right]} \tag{63}$$

$$M_V(\delta)\left(\int_{-\infty}^{\infty} \theta(s) ds \right) = \int_{-\infty}^{\infty} \theta(s) \delta(s) ds \text{ where } \delta(s+k) = \delta(s) \tag{64}$$

4. Coinsurance Arrangement Between the Underwriter and the Scheme Holder

In view of (Tse, 2009; Bass *et al.*, 2020), coinsurance deals with issues arising when insurance to value could not be achieved. The issue of underinsurance could be addressed by pricing insurance contracts to cover a fraction of severity higher than the coinsurance requirement provided the coverage conditions are

satisfied. Coinsurance clauses bind the scheme holder to preserve a specified amount of insurance in force on the insured property otherwise he faces penalty if a loss occurs. In (Tse, 2009; Bass *et al.*, 2020), the level of insurance needed is usually a computed value or a fraction of the property value. Therefore, if the policy holder buys a contract with a face value equal to or higher than the needed amount, then coinsurance will not be applicable in determining the indemnity rebate on the insured risk and hence the covered loss is fully insured beyond the deductible. But where the policyholder insures less than the required level of cover, then the degree to which coinsurance is applicable is proportional to the extent to which the schemeholder has fallen short of the requirement at the time of loss. The policy could be amended in such a way that the underwriter and the policyholder distribute the loss to themselves in a loss event where the policy holder indemnifies a certain proportion (coinsurance factor) of the loss in a loss event. The amount payable by the underwriter is

$$\mathrm{f}_{\mathrm{X}_\alpha}(x) = \frac{1}{\alpha} \mathrm{f}_{\mathrm{X}}\left(\frac{x}{\alpha}\right), 0 < \alpha < 1, Coinsurance\ factor = \alpha \text{.} \tag{65}$$

In casualty, coinsurance represents distribution of losses between the underwriter and the policy holder such that the underwriter pays a certain percentage of every of the insured loss equivalent to the coinsurance apportionment ratio. The coinsurance apportionment ratio r_{app} which is defined as $r_{app} = \dfrac{Sum\,insures\,S}{stated\ sum} = \dfrac{Sum\,insures\,S}{percentage\ of\ \text{policy value}(\alpha\rho)}$ is only applicable after the deductible modifications and other policy restrictions have been satisfied.

4.1 Theorem 1

Let $\mathrm{f}_{\mathrm{X}_\alpha}(x) = \frac{1}{\alpha} \mathrm{f}_{\mathrm{X}}\left(\frac{x}{\alpha}\right), 0 < \alpha < 1; Coinsurance\ factor CF = \alpha$, then the mean

$$\mu^{*}=\sum_{j=1}^{m}P_{j}x_{j}^{*}$$

Proof

Now, we define the expected value as follows

$$E\left(X_{\alpha}\right)=\int_{-\infty}^{\infty}xf_{X_{\alpha}}\left(x\right)dx=\frac{1}{\alpha}\int_{-\infty}^{\infty}xf_{X}\left(\frac{x}{\alpha}\right)dx \tag{66}$$

Since density is only defined on the real line, we integrate from zero to infinity

$$E\left(X_{\alpha}\right)=\int_{0}^{\infty}xf_{X_{\alpha}}\left(x\right)dx=\frac{1}{\alpha}\int_{-\infty}^{\infty}xf_{X}\left(\frac{x}{\alpha}\right)dx \tag{67}$$

$$E\left(X_{\alpha}\right)=\frac{1}{\alpha}\int_{0}^{\infty}x\sum_{j=1}^{m}P_{j}\delta\left(\frac{X}{\alpha}-x_{j}^{*}\right)dx=\frac{1}{\alpha}\sum_{j=1}^{m}P_{j}\int_{0}^{\infty}x\delta\left(\frac{X}{\alpha}-x_{j}^{*}\right)dx \tag{68}$$

$$E\left(X_{\alpha}\right)=\frac{1}{\alpha}\sum_{j=1}^{m}P_{j}\alpha x_{j}^{*}=\sum_{j=1}^{m}P_{j}x_{j}^{*}=\mu^{*} \tag{69}$$

This describes the expected claim liability under coinsurance clauses. The severity distribution describes the conditional probability distribution of losses where a loss of defined magnitude occurs. The distribution of severity defines losses as a proportion of the property value. Collecting data to model severity either as a fixed value or a fraction of the value of the asset could be very tough. The combination of loss data from non-homogeneous properties or wrong appraisal of the asset may cause misrepresentation. Loss data use the knowledge of frequency to quantify the number of times that claims occur in addition to severity. Frequency-severity factor is crucial in claim insurance modeling as a result of insurance policies and loss data profile which insurers keep. Following (Bass *et al.*, 2020; Ogungbenle *et al*, 2020), at the basic threshold, insurers accept premiums with a promise to indemnify the scheme holder in the event an insured peril occurs. This indemnity specifies the cost of claim, the severity defining the benefit outgo to the underwriter but serving as palliative measure to the policyholder. In (Ogungbenle *et al*, 2020), frequency-severity models afford the underwriter to determine the expected number of claims that the underwriter may likely experience in an insurance period and the cost of average claim. In frequency-severity technique, past data profile is used to estimate average number of claims and the average cost per claim. Following (Tse, 2009; Bass *et al*,2020), insurance experts use actuarial models to compute the probability that insurer firm will pay out a claim which will be subsequently required for underwriting decision process.

In (Schlensinger, 1981; Schlensinger, 1985; Tse, 2009; Thogersen, 2016), we can infer that an insurance contract with deductible but devoid of maximum covered loss result in sudden and extreme aggregate losses where the insured faces accumulation of aggregate severities. However, the determination of deductible tends to be very complex even with maximum covered loss. The insured further receives stop loss cover on the retained losses which consequently reduces the normal deductible. In an excess of loss contract, the underwriter consents to indemnify the insured on losses Above a predefined level C which is described as the deductible of the contract, a loss X is distributed between the insured who bears the first C value of monetary units and the underwriter who indemnifies the excess of C. In other words, the insured bears for $\min(X,C)$ while the insurance pays the excess. In view of (Bass *et al.*, 2020), the underwriter's part is defined as follows,

$$X-\min(X,C)=\begin{cases}0 & for \quad X\le C\\ X-C & for \quad X>C\end{cases}=X(I) \tag{70}$$

$$X(I)=X-\min(X,C) \tag{71}$$

Consequently, we see (71) that $E(X(I))=E(X)-E(\min(X,C))$ (72)

$$E(\min(X,C))=\int_0^C xdF_X(x)+\int_d^\infty CdF_X(x) \tag{73}$$

$$E(\min(X,C))=\int_0^C xdF_X(x)+C(S_X(C))=\int_0^C xdF_X(x)+C(S_X(C)) \tag{74}$$

Applying integration by parts letting $V=1-F_X(x)$; $U=x$; $\frac{-dV}{dx}=f_X(x)$ (75)

$$E(\min(X,C))=-x\left[1-F_X(x)\right]_0^C+\int_0^C\left[1-F_X(x)\right]dx+C\left[1-F_X(C)\right] \tag{76}$$

$$E(\min(X,C))=-C\left[1-F_X(C)\right]+\int_0^C\left[1-F_X(x)\right]dx+C\left[1-F_X(C)\right] \tag{77}$$

$$E(\min(X,C))=\int_0^C\left[1-F_X(x)\right]dx \tag{78}$$

$$E(\min(X,C))=\int_0^C S_X(x)dx \tag{79}$$

If C is the deductible and M is the maximum loss under cover with $C<M$. The loss random variable in the loss event is designated Y

$$Y=\alpha\left((X-C)_+-(X-M)_+\right) \tag{80}$$

$$E(Y)=\alpha\int_C^\infty(x-C)f_X(x)dx+\alpha\int_M^\infty(x-M)f_X(x)dx \tag{81}$$

$$E(Y)=\alpha E\left((X-C)_+\right)-\alpha E\left((X-M)_+\right) \tag{82}$$

$$E(Y^2)=E\left((X-C)_+-(X-M)_+\right)^2 \tag{83}$$

$$E(Y^2)=E\left((X-C)_+\right)^2+E\left((X-M)_+\right)^2-2E\left((X-C)_+(X-M)_+\right) \tag{84}$$

$$E\left(Y^2\right)=\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+C^2\left[S_X(C)\right]+\int_0^M x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+M^2\left[S_X(M)\right]-$$
$$2\left[\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+\mathrm{CM}S_X(M)-C\int_C^M x\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}\right] \tag{85}$$

$$E\left(Y^2\right)=\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+C^2\left[S_X(C)\right]+\int_0^M x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+M^2\left[S_X(M)\right]-$$
$$2\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}-2\mathrm{CM}S_X(M)+2C\int_C^M x\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx} \tag{86}$$

$$E\left(Y^2\right)=-\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+C^2\left[S_X(C)\right]+\int_0^M x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+M^2\left[S_X(M)\right]-$$
$$2\mathrm{CM}S_X(M)+2C\int_C^M x\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx} \tag{87}$$

$$Var\left(Y^2\right)=-\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+C^2\left[S_X(C)\right]+\int_0^M x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+M^2\left[S_X(M)\right]-$$
$$2\mathrm{CM}S_X(M)+2C\int_C^M x\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}-\left[\alpha E\left((X-C)_+\right)-\alpha E\left((X-M)_+\right)\right]^2 \tag{88}$$

$$Var\left(Y^2\right)=-\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+C^2\left[S_X(C)\right]+\int_0^M x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+M^2\left[S_X(M)\right]-$$
$$2\mathrm{CM}S_X(M)+2C\int_C^M x\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}-\left[\alpha E\left((X-C)_+\right)\right]^2-\left[\alpha E\left((X-M)_+\right)\right]^2+$$
$$2\alpha^2 E\left((X-C)_+\right)E\left((X-M)_+\right) \tag{89}$$

$$\int_{x_1}^{x_2}\delta(x-t_0)\,f(x)\,dx=\int_{x_1}^{x_2}\delta(x-t_0)\,f(t_0)\,dx=f(t_0)\int_{x_1}^{x_2}\delta(x-t_0)\,dx,\; x_1<t_0<x_2 \tag{90}$$

$$Var\left(Y^2\right)=-\int_0^C x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+C^2\left[S_X(C)\right]+\int_0^M x^2 \mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}+M^2\left[S_X(M)\right]-$$
$$2\mathrm{CM}S_X(M)+2C\int_C^M x\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}-\alpha^2\left[\int_C^\infty (X-C)\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}\right]^2-\alpha^2\left[\int_M^\infty (X-M)\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}\right]^2+$$
$$2\alpha^2\int_C^\infty (X-C)\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx}\int_M^\infty (X-M)\mathrm{f}_{\mathrm{X}}(x)\mathrm{dx} \tag{91}$$

$$Var\left(Y^2\right)=-\int_0^C x^2\sum_{j=1}^{m}P_j\delta\left(X-x_j^*\right)dx+C^2\left[S_X\left(C\right)\right]+\int_0^M x^2\sum_{j=1}^{m}P_j\delta\left(X-x_j^*\right)dx+M^2\left[S_X\left(M\right)\right]- 2CMS_X\left(M\right)+2C\int_C^M x\sum_{j=1}^{m}P_j\delta\left(X-x_j^*\right)dx-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-C\right]^2-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-M\right]^2+ 2\alpha^2\left(\sum_{j=1}^{m}P_jx_j^*-C\right)\left(\sum_{j=1}^{m}P_jx_j^*-M\right) \quad (92)$$

$$Var\left(Y^2\right)=-\sum_{j=1}^{m}P_j\int_0^C x^2\delta\left(X-x_j^*\right)dx+C^2\left[S_X\left(C\right)\right]+\sum_{j=1}^{m}P_j\int_0^M x^2\delta\left(X-x_j^*\right)dx+M^2\left[S_X\left(M\right)\right]- 2CMS_X\left(M\right)+2C\sum_{j=1}^{m}P_j\int_C^M x\delta\left(X-x_j^*\right)dx-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-C\right]^2-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-M\right]^2+ 2\alpha^2\left(\sum_{j=1}^{m}P_jx_j^*-C\right)\left(\sum_{j=1}^{m}P_jx_j^*-M\right) \quad (93)$$

$$Var\left(Y^2\right)=-\sum_{j=1}^{m}P_j\int_0^C x_j^{*2}\delta\left(X-x_j^*\right)dx+C^2\left[S_X\left(C\right)\right]+\sum_{j=1}^{m}P_j\int_0^M x_j^{*2}\delta\left(X-x_j^*\right)dx+M^2\left[S_X\left(M\right)\right]- 2CMS_X\left(M\right)+2C\sum_{j=1}^{m}P_j\int_C^M x_j^*\delta\left(X-x_j^*\right)dx-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-C\right]^2-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-M\right]^2+ 2\alpha^2\left(\sum_{j=1}^{m}P_jx_j^*-C\right)\left(\sum_{j=1}^{m}P_jx_j^*-M\right) \quad (94)$$

$$Var\left(Y^2\right)=-\sum_{j=1}^{m}P_jx_j^{*2}\int_0^C \delta\left(X-x_j^*\right)dx+C^2\left[S_X\left(C\right)\right]+\sum_{j=1}^{m}P_jx_j^{*2}\int_0^M \delta\left(X-x_j^*\right)dx+M^2\left[S_X\left(M\right)\right]- 2CMS_X\left(M\right)+2C\sum_{j=1}^{m}P_jx_j^*\int_C^M \delta\left(X-x_j^*\right)dx-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-C\right]^2-\alpha^2\left[\sum_{j=1}^{m}P_jx_j^*-M\right]^2+ 2\alpha^2\left(\sum_{j=1}^{m}P_jx_j^*-C\right)\left(\sum_{j=1}^{m}P_jx_j^*-M\right) \quad (95)$$

4.2 Theorem 2

Suppose X is a continuous random insurable risk with density $g_X\left(x\right)$ and C is a deductible, then

$$E\left(\left(X-C\right)_+\right)=\int_C^\infty\left(\int_x^\infty g\left(u\right)\right)dx \quad (96)$$

Proof

$$\left(X-C\right)_+ > x, iff, X-C>x \Rightarrow X>C+x \quad (97)$$

$$Pr\left(\left(X-C\right)_+ > x\right)+Pr\left(\left(X-C\right)_+ < x\right)=\int_0^\infty g\left(u\right)du=1 \quad (98)$$

$$Pr\left(\left(X-C\right)_{+}>x\right)+\int_{-\infty}^{X+C}g_{U}\left(u\right)du=1 \tag{99}$$

$$Pr\left(\left(X-C\right)_{+}>x\right)=1-\int_{-\infty}^{X+C}g_{U}\left(u\right)du \tag{100}$$

$$E\left(\left(X-C\right)_{+}\right)=\int_{0}^{\infty}\left(1-\int_{-\infty}^{X+C}g_{U}\left(u\right)du\right)dx \tag{101}$$

$$let, s=u-C \tag{102}$$

$$E\left(\left(X-C\right)_{+}\right)=\int_{C}^{\infty}\left(1-\int_{-\infty}^{x}g_{S}\left(s\right)ds\right)dx=\int_{C}^{\infty}\left(\int_{x}^{\infty}g_{S}\left(s\right)ds\right)dx \tag{103}$$

$$E\left(\left(X-C\right)_{+}\right)=\int_{C}^{\infty}\left(1-\int_{-\infty}^{x}g_{S}\left(s+C\right)ds\right)dx=\int_{C}^{\infty}\left(\int_{x}^{\infty}g_{S}\left(s\right)ds\right)dx \tag{104}$$

Furthermore, $\int_{-\infty}^{\infty}g_{S}\left(s+C\right)ds=\int_{-\infty}^{x}g_{S}\left(s+C\right)ds+\int_{x}^{\infty}g_{S}\left(s+C\right)ds=1$ (105)

$$E\left(\left(X-C\right)_{+}\right)=\int_{C}^{\infty}\left(x-c\right)g_{X}\left(x\right)dx \tag{106}$$

4.2 Numerical Illustration on Expected Loss

Suppose $X\sim EXP\left(\theta\right),\theta=1,C=0.30,M=6,\alpha=0.90$

$$\left\langle X_{L}\right\rangle=\int_{0.30}^{\infty}e^{-x}dx=e^{-0.30}=1.349858808 \tag{107}$$

$$E\left[\left(X-M\right)_{+}\right]=\int_{M}^{\infty}e^{-x}dx=e^{-6}=0.002478752177 \tag{108}$$

$\left\langle Y\right\rangle=0.90\left(1.349858808-0.002478752177\right)=0.90\left(1.347380056\right)=1.2126$. This is the expected loss in a loss event.

Discussion of Results

Let $f_{X_{\alpha}}\left(x\right)=\frac{1}{\alpha}f_{X}\left(\frac{x}{\alpha}\right),0<\alpha<1; Coinsurance\ factor CF=\alpha$ **,** then (109)

Recall in (4d) that, the probability density is defined by

$$f_{X}\left(\theta\right)=E_{x}\left(I_{0<x\leq\theta}\right)=E_{x}\left(H\left(\theta-x\right)\right)=h_{x}\left(\theta\right)$$

$$\mathrm{E}\left(\mathrm{X}_{\alpha}\right)=\int_{-\infty}^{\infty}x\mathrm{f}_{\mathrm{X}_{\alpha}}\left(\mathrm{x}\right)\mathrm{dx}=\frac{1}{\alpha}\int_{-\infty}^{\infty}x\mathrm{f}_{\mathrm{X}_{\alpha}}\left(\mathrm{x}\right)\mathrm{dx} \tag{110}$$

Since density is only defined on the real line, we integrate from zero to infinity

$$\mathrm{E}\left(\mathrm{X}_\alpha\right)=\int_0^\infty x\mathrm{f}_{\mathrm{X}_\alpha}\left(\mathrm{x}\right)\mathrm{dx}=\frac{1}{\alpha}\int_{-\infty}^{\infty} xE_x\left(I_{0<x\le\frac{\theta}{\alpha}}\right)\mathrm{dx} \tag{111}$$

$$\mathrm{E}\left(\mathrm{X}_\alpha\right)=\int_0^\infty x\mathrm{f}_{\mathrm{X}_\alpha}\left(\mathrm{x}\right)\mathrm{dx}=\frac{1}{\alpha}\int_{-\infty}^{\infty} xE_x\left(H\left(\frac{\theta}{\alpha}-x\right)\right)\mathrm{dx} \tag{112}$$

from equation (4), we have

$$\mathrm{E}\left(\mathrm{X}_\alpha\right)=\int_0^\infty x\mathrm{f}_{\mathrm{X}_\alpha}\left(\mathrm{x}\right)\mathrm{dx}=\frac{1}{\alpha}\int_{-\infty}^{\infty} xh_x\left(\frac{\theta}{\alpha}\right)\mathrm{dx} \tag{113}$$

Furthermore, the second moment is $\mathrm{E}\left(\mathrm{X}^2{}_\alpha\right)=\int_{-\infty}^{\infty} x^2\mathrm{f}_{\mathrm{X}_\alpha}\left(\mathrm{x}\right)\mathrm{dx}=\frac{1}{\alpha}\int_{-\infty}^{\infty} x^2\mathrm{f}_{\mathrm{X}}\left(\frac{x}{\alpha}\right)\mathrm{dx}$ (114)

Since density is only defined on the real line, we integrate from zero to infinity

$$\mathrm{E}\left(\mathrm{X}^2{}_\alpha\right)=\int_0^\infty x^2\mathrm{f}_{\mathrm{X}_\alpha}\left(\mathrm{x}\right)\mathrm{dx}=\frac{1}{\alpha}\int_0^\infty x^2\mathrm{f}_{\mathrm{X}}\left(\frac{x}{\alpha}\right)\mathrm{dx} \tag{115}$$

$$\mathrm{E}\left(\mathrm{X}^2{}_\alpha\right)=\frac{1}{\alpha}\int_0^\infty x^2\sum_{\mathrm{j}=1}^{\mathrm{m}}\mathrm{P_j}\delta\left(\frac{x}{\alpha}-\mathrm{x_j}^*\right)\mathrm{dx}=\frac{1}{\alpha}\sum_{j=1}^{m}P_j\int_0^\infty x^2\delta\left(\frac{x}{\alpha}-\mathrm{x_j}^*\right)dx \tag{116}$$

$$\mathrm{E}\left(\mathrm{X}^2{}_\alpha\right)=\frac{1}{\alpha}\sum_{j=1}^{m}P_j\int_0^\infty x^2\delta\left(\frac{x}{\alpha}-\mathrm{x_j}^*\right)dx \tag{117}$$

$$\mathrm{E}\left(\mathrm{X}^2{}_\alpha\right)=\frac{1}{\alpha}\sum_{\mathrm{j}=1}^{\mathrm{m}}\mathrm{P_j}\alpha^2\mathrm{x_j}^{*2}=\alpha\sum_{\mathrm{j}=1}^{\mathrm{m}}\mathrm{P_j}\mathrm{x_j}^{*2}=\frac{\alpha}{\mathrm{P_j}}\sum_{\mathrm{j}=1}^{\mathrm{m}}\mathrm{P}_j^2\mathrm{x_j}^{*2}=\frac{\alpha}{\mathrm{P_j}}\mu^{*2} \tag{118}$$

$$\mathrm{E}\left(\mathrm{X^n}{}_\alpha\right)=\frac{1}{\alpha}\sum_{\mathrm{j}=1}^{\mathrm{m}}\mathrm{P_j}\alpha^n\mathrm{x_j}^{*\mathrm{n}}=\alpha^{n-1}\sum_{\mathrm{j}=1}^{\mathrm{m}}\mathrm{P_j}\mathrm{x_j}^{*\mathrm{n}}=\frac{\alpha^{n-1}}{\mathrm{P}_j^{n-1}}\mu^{*n} \tag{119}$$

The variance $\mathrm{Var}\left(\mathrm{X}_\alpha\right)=\mathrm{E}\left(\mathrm{X}^2{}_\alpha\right)-\left(\mathrm{E}\left(\mathrm{X}_\alpha\right)\right)^2$ (120)

$$\mathrm{Var}\left(\mathrm{X}_\alpha\right)=\frac{\alpha}{\mathrm{P_j}}\mu^{*2}-\left(\mu^*\right)^2=\mu^*\left(\frac{\alpha}{\mathrm{P_j}}\mu^*-\mu^*\right) \tag{121}$$

We see that in the event that $\alpha=\mathrm{P_j}$, then the variance will be zero

5. Conclusion

A discrete mass probability function can be converted to a continuous density distribution through the application of a sequence of dirac-delta functions in such a way that the discrete random risk X of $(k+1)$ discrete mass points x_n; $n=0,1,2,3,....k$ each with probability α_n have generalized distribution defined as follows $\varphi_X(x)=\sum_{n=0}^{k}\alpha_n\delta(x-x_n)$. If $f(x)$ is some function, then the expectation

$$E_X\left(f\left(x\right)\right)=\int_{-\infty}^{\infty}f\left(x\right)\theta_X\left(x\right)dx=\sum_{n=0}^{k}\alpha_n\int f\left(x\right)\delta\left(x-x_n\right)dx=\sum_{n=0}^{k}\alpha_n f\left(x_n\right)$$

In this paper, interesting properties of dirac-delta function has been shown as appropriate in coinsured business. We obtained some results on insurance claims size of significant application in general insurance practice thereby improving on the deterministic actuarial literature. The precise estimation of frequency and severity of insurance claims permits an underwriter to satisfy claims liability as they come and meet solvency conditions. This paper has displayed interesting actuarial model of deep interest to insurance practice which was achieved under defined actuarial assumptions. In this paper, we have demonstrated how dirac-delta has been applied to formulate actuarial density of a random risk. Thus, in explaining a unified ground of applying specialized functions to investigate the behavior of risk functions, the singularity potential method was applied to investigate expected loss in a coinsured business regarding claim severities, the variance function and complex risk. The rationale behind employing singularity potential function, is characterized in its superiority to permit alternative method to attain analytically useful models for coinsured business severity function. In this paper, we have used the dirac-delta function to determine.

- The expected cost per loss claim severity under coinsurance arrangement with deductible restrictions
- The second moment of cost per loss claim severity under coinsurance arrangement with deductible restrictions and policy limit
- The variance of the cost per payment loss event under coinsurance arrangement with deductible restrictions
- $\theta_{X+Y}(s) = \theta_Y(s)\theta_X(s)$ under independence of insurance risk

Numerical results on coinsured business technique under mathematical real analysis framework will be pursued in future research.

References

1. Bass R.F., Ruiz P.A., Baudoin F., Gordina M., Mariano P., Mostovyi O., Sengupta A., Teplyaev A., and Valdez E. (2020). Upper-level undergraduate probability with actuarial and financial applications. Department of mathematics, University of Connecticut.
2. https://www.google.com/search?q=conditional+expectation+in+actuarial+mathematics+pdf
3. Chakraborty S. (2008). Some applications of dirac-delta function in statistics for more than one
4. random variable, Application and Applied Mathematics Journal 3(1), 42-54.
5. Das A., and Melissinos A.C. (1986). Quantum mechanics: A modern introduction. Gorden and
6. Breach Science Publishers.
7. Kanwal R.P. (1998). Function theory and technique (2nd ed), Birkhauser, Boston,1998, pp.1-17.
8. Khuri A.I. (2004). Applications of dirac's delta function in statistics. International Journal of Mathematical Education in Science and Technology, 35(2), 185-195.
9. Mohammed H.R. (2011). Dirac's function and its application in solving some problems in mathematics. Journal of Al-Qadisiya for Computer Science and Mathematics 3, 1-15.
10. Ogungbenle, M. G., Adeyele, J.S., and Mesioye A.E. (2020). Modeling moments of insurance claim size under dirac-delta function. Daffodil International University Journal of Science and Technology, 15(1), 43-51.
11. Ogungbenle, M.G., Kusa, N.D., and Adeoye, S. V. (2021). A computational analysis of singularity

12. potential induced characterisation of higher moments on insurance claim size in a loss event. Journal of Science, technology, Mathematics, and Education, Federal University of Technology, Minna. Forthcoming, Accepted for Publication.
13. Onural L. (2006). Impulse functions over curves and surfaces and their application to diffraction. Journal of Mathematical Analysis and Applications 322(1), 18-27.
14. Pazman A., and Pronzato L. (1996). A dirac-function method for densities of nonlinear statistics
15. and for marginal densities in nonlinear regression. Statistics & Probability Letters, 26,159-167.
16. Dirac P.A.M. (1985). The principles of quantum mechanics, Oxford University Press, IV edition.
17. Salasnich L. (2014). Quantum physics of light and matter, unitext for physics, Springer Publishing International, Switzerland, [Online]. Available: http://cds.cern.ch/record/1707567/files/978-3-319-05179-6_BookBackMatter.pdfpp.179-195, [Assessed Aug. 2, 2019]
18. Sastry S.S. (2009). Advanced engineering mathematics. PHI learning ltd, New Delhi, 2009, pp. 36-57.
19. Schlesinger H. (1981). The optimal level of deductivity in insurance contracts. Journal of Risk and Insurance. 48(3), 465-481.
20. Schlesinger H. (1985). Choosing a deductible for insurance contracts: Best or worst insurance policy?'', Journal of Risk and Insurance. 52(3), 522-527.
21. Thogersen J. (2016). Optimal premium as a function of the deductible: Customer analysis and portfolio characteristics. Risks 4, 1-19.
22. Tse Y.K. (2009). Non-life actuarial models, Cambridge University Press, U. K, pp. 66-75.
23. Zhang L. (2018). On the calculus of dirac-delta function with some applications. International
24. Journal of Mathematics Trends and Technology. 56(4), 258-270.